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A Monte Carlo Method for Sensitivity Analysis and Parametric Optimization of Nonlinear Stochastic Systems: The Ergodic Case.

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Abstract

For high dimensional or nonlinear problems there are serious limitations on the power of available computational methods for the optimization or parametric optimization of stochastic systems of diffusion type. The paper develops an effective Monte Carlo method for obtaining good estimators of systems sensitivities with respect to system parameters, when the system is of interest over a long period of time. The value of the method is borne out by numerical experiments, and the computational requirements are favorable with respect to competing methods when the dimension is high or the nonlinearities 'severe'. The method is a type of "derivative of likelihood ratio" method. For a wide class of problems, the cost function or dynamics need not be smooth in the state variables; for example, where the cost is the probability of an event or "sign" functions appear in the dynamics. Under appropriate conditions, it is shown that the invariant measures are differentiable with respect to the parameters. Since the basic diffusion (or other) model cannot be simulated exactly, simulatable approximations are discussed in detail, and estimators of the derivatives of the cost functions for these approximations are obtained and analyzed. It is shown that these estimators and their expectations converge to those for the original problem. Thus, we prove a robustness result for the sensitivity estimators, namely that the derivatives of the ergodic cost functions (and their estimators) for the simulatable approximations converge to those for the approximated process. Such results are essential if a simulation based method is to be used with confidence.

Key words: Monte Carlo method for diffusions, parametric optimization of stochastic systems, sensitivity analysis, optimization of stochastic systems, nonlinear stochastic systems, high dimensional stochastic systems, parametric optimization of diffusion processes, likelihood ratio method for sensitivity analysis, parametric derivatives of invariant measures, ergodic control.

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1. Introduction

This paper is concerned with a key question in the use of recursive Monte Carlo methods for system optimization, when the system operation and cost are of interest for a long period of time. For many control systems, the control is given a-priori in a parametrized form and for the use of Monte Carlo methods for the optimization of the parameter, one needs good estimators of the derivatives of the cost function with respect to the parameter.

Reference [1] develops a very useful method for doing this, when the system is of the diffusion or related type, and the control interval of concern is finite. Numerical approximations to the unbiased estimators were developed and analyzed, and simulations showed that the method can be superior to competing methods if the system dimension is large or the system nonlinear. In this paper, the results of [1] are extended to the ergodic cost problem. New difficulties arise, since we need essentially to deal with derivatives of the invariant measures with respect to the control parameters and with the convergence of suitable computable approximations. Owing to these "ergodic" problems, the assumptions are stronger here than in [1].

Let $x(\cdot)$ be defined by the diffusion

$$dx = b(x,\alpha)dt + \sigma(x)dw, \quad x \in R^r, \tag{1.1}$$

where $a(x) = \sigma(x)\sigma'(x)$ is non-degenerate and α is a control parameter to be chosen. For each α of interest, let $x(\cdot)$ have a unique invariant measure $\mu(\alpha)$. Precise conditions will be given below. For 'smooth cost rate' $k(\cdot)$, define the "ergodic cost"

$$\langle \mu(\alpha), k(\alpha)\rangle \equiv \int \mu(dx,\alpha)k(x,\alpha) \equiv \bar{k}(\alpha). \tag{1.2}$$

We wish to get an unbiased estimator of $\partial\bar{k}(\alpha)/\partial\alpha$ (as well as reasonable 'numerical' approximations from sample simulations) at selected values of α. Such estimators are necessary if we wish to minimize $\bar{k}(\alpha)$ over α by some recursive Monte Carlo (stochastic approximation) method.

Control problems are frequently of this type; i.e., the control is given in a parametric form. Often, a full optimal feedback control is not desired since it might be very hard to implement and all the state variables are not available. But a good class of parametrized controls might be known. See [1] for some examples and further motivation, as well as a discussion of alternative approaches.

Generally, one cannot easily evaluate $\bar{k}(\alpha)$ or its derivatives. Then one might seek a method for getting good estimators which can be used in a recursive Monte Carlo optimization method. The ease of getting the estimates and their quality are key issues in such an approach. The estimators are to be obtained by simulations of (1.1) or of approximations to (1.1), since the solution of (1.1) can not be known exactly.

Reference [1] developed a general "likelihood ratio derivative" based method for getting such estimators, under conditions which are much broader than those used in this paper, but for a 'finite time' problem. The numerical data in [1], and that obtained subsequently, show that the method can be quite superior to its competitors for non-linear and high dimensional systems. The quality of the estimator is judged by the "variance per CPU time required." The reader is referred to [1] for more motivation and examples. The ergodic cost problem is harder and requires stronger (hence, the non-degeneracy) conditions. Actually, the method has been successfully tested on many degenerate problems of the type used in [1]. so that the conditions which our analysis requires can undoubtedly be weakened. There are ready extensions to the jump-diffusion, reflection and other standard models. In order to introduce the idea, we give a brief informal review of one idea in [1], but using our slightly different terminology, and under stronger conditions than used in [1].

For given $T < \infty$, define the "finite time" costs

$$C(x, \alpha) = \int_0^T k(x(s), \alpha) ds + k_0(x(T), \alpha),$$
$$\bar{C}(x, \alpha) = E_x^\alpha C(x, \alpha),$$

where E_x^α denotes the expectation with parameter α and $x(0) = x$. We always use α_0 to denote the point at which the derivative is to be taken. With no loss of generality α will be a real number, since for the vector case we can estimate the derivative for each component separately. Let $P_x^\alpha(T)$ denote the measure induced by the solution to (1.1) with the initial condition $x(0) = x$, on $C^r[0, T]$, the space of R^r-valued continuous functions on $[0, T]$, with the sup norm. Let $b(x, \alpha), k(x, \alpha)$ and $k_0(x, \alpha)$ be α-differentiable and define $\alpha = \alpha_0 + \delta\alpha$ and $\delta b(x, \alpha_0, \delta\alpha) = b(x, \alpha_0 + \delta\alpha) - b(x, \alpha_0)$. Define

$$\xi(0, T; \alpha_0, \delta\alpha) = \int_0^T [\sigma^{-1}(x(s))\delta b(x(s), \alpha_0, \delta\alpha)]' dw(s)$$
$$-\frac{1}{2}\int_0^T |\sigma^{-1}(x(s))\delta b(x(s), \alpha_0, \delta\alpha)|^2 ds,$$

and the Radon–Nikodym derivative

$$\frac{dP_x^{\alpha_0+\delta\alpha}(T)}{dP_x^{\alpha_0}(T)} = \exp \xi(0, T; \alpha_0, \delta\alpha). \tag{1.3}$$

Define $Z(\cdot)$ by

$$Z(T, \alpha_0) = \int_0^T [\sigma^{-1}(x(s)) b_\alpha(x(s), \alpha_0)]' dw(s) \tag{1.4}$$

$$= \int_0^T [b'_\alpha(x(s),\alpha_0)a^{-1}(x(s))][dx(s) - b(x(s),\alpha_0)ds].$$

We use the subscripted $b_\alpha(x,\alpha_0)$, etc., to denote the α-derivatives at α_0. Then the quantities

$$(1.5) \qquad Q(\alpha_0) = \int_0^T [k(x(s),\alpha_0)Z(s,\alpha_0) + k_\alpha(x(s),\alpha_0)]ds$$
$$+ k_0(x(T),\alpha_0)Z(T,\alpha_0) + k_{0,\alpha}(x(T),\alpha_0),$$

$$(1.5') \qquad \hat{Q}(\alpha_0) = \int_0^T \left[(k(x(s),\alpha_0) - \bar{k}(x(s),\alpha_0))Z(s,\alpha_0) + k_\alpha(x(s),\alpha_0)\right]ds$$
$$+ (k_0(x(T),\alpha_0) - \bar{k}_0(x(T),\alpha_0))Z(T,\alpha_0) + k_{0,\alpha}(x(T),\alpha_0),$$

where we use

$$\bar{k}(x(s),\alpha_0) = E_x^{\alpha_0} k(x(s),\alpha_0),$$

are unbiased estimators of $\bar{C}_\alpha(x,\alpha_0)$. Thus, if a path of $x(\cdot)$ is available, one can calculate or approximate (1.5) or (1.5′).

In order to avoid the very time consuming task of evaluating (from the simulations) $\bar{k}(x(s),\alpha_0)$ for each $s \le T$, in (1.5′), we usually use $\bar{k}(x(T),\alpha_0)$ in place of $\bar{k}(x(s),\alpha_0)$, and with good results.

Generally, paths of the true model $x(\cdot)$ are not available, and one can only approximate via a numerical method (say, a discrete time approximation). Reference [1] discusses two basic classes of such approximations and proves that the estimators obtained from them are good. Getting good estimators is more difficult for the ergodic problem, since we also need to truncate the infinite time interval and approximate (at least implicitly) derivatives of invariant measures, a non-trivial problem.

The proofs use a representation of the invariant measure of the diffusion process in terms of that of an imbedded Markov chain, defined by the random return times to a "recurrence set", as well as certain Girsanov transformations defined on these "return intervals". In order to be sure that these transformations are well defined, a bound on an exponential moment of the return time is needed. This is provided by the stability result in Section 2. Section 3 is concerned with ergodic properties of the diffusion model. The imbedded Morkov chain is defined, and the invariant measure of the diffusion is defined in terms of this Markov chain, and the needed recurrence (*ϕ-recurrence*) properties of the chains are stated. Section 4 is concerned with the existence of the derivative of the invariant measure of the diffusion with respect to the parameter. The differentiability is first shown for the invariant measure of the imbedded chain, and then this is used to get the result for the diffusion. The differentiability is in two senses, setwise convergence and weak convergence. Some preliminary results concerning equicontinuity of certain sets of functions and invertability of the operator $I - \tilde{P}(\alpha_0)$ (defined in the section) are first proved. It is also shown that the derivative of the invariant measure can be well approximated by the derivative of the transition function for large enough time.

Since the diffusion model is an "ideal" model and the paths can at best be approximated in some statistical sense, one needs to know that the natural approximations can be used with confidence in any implementation. Reference [1] dealt with two types of approximations, a discrete time model and a Markov chain approximation. Either can be used here, but we restrict our attention to the first approximation. The model is introduced in Section 5, and some preliminary sensitivity results are stated there. Some needed stability estimates (analogous to the estimates of Section 2), uniform in the approximation parameter, are obtained in Section 6. The main theoretical results for the approximations are in Section 7, where, after getting some preliminary results concerning the rate of convergence of certain quantities to their "invariant means", it is shown that the invariant measure of the discrete time approximation is differentiable with respect to the control parameter, that the derivatives converge to the derivative of the invariant measure of the diffusion, as well as results concerning finite time approximations. The results imply an important robustness of the derivatives with respect to the model. This is a new result and a very useful one from the point of view of applications, since otherwise general results concerning the existence of the derivatives for the ideal model would not have much practical relevance.

Numerical data is given in Section 8. The basic method of implementation requires the use of a discrete parameter approximation, over a finite time period. The period needs to be large enough to capture the "ergodic effects". Two methods are compared; a finite difference method, which has been altered to be fairly efficient, and several forms of our method. The comparison depends on the problem, but it is clear that for a large class of nonlinear problems, our method is preferable. One should note that reasonable examples can be constructed so that any chosen method works best, so that one needs to keep an open mind in any application.

The analysis has been restricted to nondegenerate diffusion models, but a similar analysis can be carried out with various related process, provided only that ergodic results analogous to those of Section 3 are available.

2. Stability of $x(\cdot)$

In order to develop the ergodic results and use a Girsanov measure transformat' .ı method on random unbounded intervals, suitable stability properties of $x(\cdot)$ need to be proved. We will use the following assumptions. The parameter α will be confined to a compact interval A_0 with α_0 in its interior.

A2.1. *$b(\cdot)$ and $\sigma(\cdot)$ are continuous, $\sigma(\cdot)$ is bounded and $\sigma(x)\sigma'(x) = a(x) \geq \varepsilon_0 I$ for some $\varepsilon_0 > 0$. For some $K < \infty$, $|b(x,\alpha)| \leq K|x| + K$.*

A2.2. (1.1) *has a unique weak sense solution for each $x(0) = x$ and $\alpha \in A_0$.*

A2.3. *There is a twice continuously differentiable Liapunov function $0 \leq V(x) \to \infty$ as $|x| \to \infty$ and $\varepsilon_1 > 0$ such that*

(a) *$V_{xx}(x)$ is bounded and continuous,*

(b) *$V_x'(x)b(x,\alpha) \leq -\varepsilon_1 < 0$ for large $|x|$, $\alpha \in A_0$,*

(c) $\overline{\lim}_{|x|\to\infty} \sup_{\alpha \in A_0} |V_x(x)|^2/|V_x'(x)b(x,\alpha)| < \infty$,

(d) $\overline{\lim}_{|x|\to\infty} \sup_{\alpha \in A_0} |V_{xx}(x) \cdot a(x)|/|V_x'(x)b(x,\alpha)| < 2$.

A2.4. *When $b(x,\alpha) \equiv 0$,* (1.1) *has a unique weak sense solution for each $x = x(0)$.*

A2.5. *There is a bounded continuous function $b_\alpha(\cdot,\alpha_0)$ such that as $\delta\alpha \to 0$*

$$\delta b(x,\alpha_0,\delta\alpha)/\delta\alpha \to b_\alpha(x,\alpha_0)$$

boundedly, and uniformly on each compact x-set.

Remark on (A2.3). The condition does not seem to be very restrictive. It holds, in particular, for the linear case $b(x,\alpha) = A(\alpha)x$, where $A(\alpha)$ is 'uniformly stable' for $\alpha \in A_0$.

Remark on (A2.2). (A2.4) and the stability Theorem 2.1 imply (A2.2), but it is useful to isolate it as a separate condition.

Theorem 2.1. *Assume* (A2.1)–(A2.3). *There is a compact set Q which is the closure of its interior such that for each compact $Q_1 \supset Q$ and τ_1 defined by* $\tau_1 = \min\{t : x(t) \in Q\}$, *we have for small* $\rho > 0$

$$\sup_{\alpha \in A_0} \sup_{x \in Q_1 - Q} E_x^\alpha \exp \rho\tau_1 < \infty. \tag{2.1}$$

Proof. Let $\mathcal{L}$ denote the differential generator of $x(\cdot)$: $\mathcal{L}f(x) = f_x'(x)b(x,\alpha) + \frac{1}{2}\text{trace}\, f_{xx}(x) \cdot a(x)$. Then

$$\begin{aligned} \mathcal{L}e^{\rho V(x)} = \rho e^{\rho V(x)} \big[& V_x'(x)b(x,\alpha) \\ & + \rho\, \text{trace}(V_x(x)V_x'(x)) \cdot a(x)/2 + \text{trace}\, V_{xx}(x) \cdot a(x)/2 \big]. \end{aligned}$$

Let Q be large enough and ρ small enough such that for $x \notin Q$ (use (A2.3)) and some $\lambda > 0$,

$$\mathcal{L}e^{\rho V(x)} \leq -\rho\lambda e^{\rho V(x)}. \tag{2.2}$$

It then follows that for small ρ and $x \notin Q$

$$\mathcal{L}\big[e^{\lambda\rho t} e^{\rho V(x)}\big] \leq 0. \tag{2.3}$$

From (2.3), Itô's Lemma and a stopping time argument it follows that

$$E_x^\alpha e^{\lambda\rho\tau_1} \leq E_x^\alpha e^{\lambda\rho\tau_1} e^{\rho V(x(\tau_1))} \leq e^{\rho V(x)} \tag{2.4}$$

for small ρ and $x = x(0) \notin Q$, which yields the result. Q.E.D.

Corollary. *Assume* (A2.1)–(A2.3). *Let Q and Q_1 be as in the theorem. Define τ to be the first return time of $x(\cdot)$ to Q after hitting ∂Q_1. Then, for* *small* $\rho > 0$

$$\sup_{\alpha \in A_0} \sup_{x \in \partial Q} E_x^\alpha e^{\rho\tau} < \infty. \tag{2.5}$$

The proof follows from the theorem and the non-degeneracy and is omitted.

3. Ergodic Properties of (1.1)

By (A2.1)-(A2.3) and Theorem 2.1, for each $\alpha \in A_0$, $x(\cdot)$ is a recurrent strong Feller process. Let $P(x,t,A \mid \alpha)$ denote the transition function. By [2], [3], there is a unique invariant measure $\mu(\alpha)$ with $\mu(R^r,\alpha)$ and $P(x,t,A \mid \alpha) \xrightarrow{t} \mu(A,\alpha)$ as $t \to \infty$, for each Borel A. For $t > 0$, $P(x,t,\cdot \mid \alpha)$ has a bounded and nowhere zero density with respect to Lebesgue measure and so does $\mu(\alpha)$.

We next state a representation of $\mu(\alpha)$ first used by Khazminskii [2] and which is very useful for analysis. The representation is useful largely because it is hard to work with ergodic problems and to deal with questions concerning convergence to invariant measures when the state space is unbounded.

Let $G_1 \supset G$ be compact sets, each of which is connected and is the closure of its interior. Denote the boundaries by Γ_1 and Γ, resp., and let G be strictly interior to G_1. Let Γ and Γ_1 be differentiable. Define the stopping times:

$$\tau' = \inf\{t : x(t) \in \Gamma_1\}$$

$$\tau_1 = \inf\{t : x(t) \in \Gamma\},$$

$$\tau_1' = \inf\{t > \tau_1 : x(t) \in \Gamma_1\}.$$

For $n > 1$,

$$\tau_n = \inf\{t > \tau_{n-1}' : x(t) \in \Gamma\},$$

$$\tau_n' = \inf\{t > \tau_n : x(t) \in \Gamma_1\}.$$

For $x = x(0) \in \Gamma$, we use τ to denote $\tau_2 - \tau_1 = \tau_2$, the canonical "return" time to Γ.

By Theorem 2.1, for small $\rho > 0$,

$$\sup_{x\in\Gamma,\alpha\in A_0} E_x^\alpha \tau < \infty, \qquad \sup_{x\in\Gamma,\alpha\in A_0} E_x^\alpha e^{\rho\tau} < \infty. \tag{3.1}$$

Let $\alpha \in A_0$. Define the process $\tilde{X}_n = x(\tau_n)$. By [2] and (A2.1)–(A2.3), $\{\tilde{X}_n\}$ is a recurrent homogeneous Markov chain on Γ. Let $\tilde{P}(x, n, \cdot \mid \alpha)$ denote its transition probability. It has a unique invariant measure $\tilde{\mu}(\alpha)$.

The chain is also defined for initial condition $x = \tilde{X}_0 \in G$. Even though $\tilde{X}_n \in \Gamma$, for $n \geq 1$, it will be useful to use G as the state space in Section 6 and afterwards in order to unify the notation with that for the approximations. The results up to Section 5 hold with this change.

Define $\tau(A) = \int_0^\tau I_A(x(s))ds$ for Borel sets A. Then we can write [2,3]

$$\mu(A, \alpha) = \bar{\mu}(A, \alpha)/\bar{\mu}(R^r, \alpha), \tag{3.2}$$

where

$$\bar{\mu}(A, \alpha) = \int_\Gamma \tilde{\mu}(dx, \alpha) E_x^\alpha \tau(A).$$

Hence, for bounded measurable $f(\cdot)$, we have the representation

$$\langle \mu(\alpha), f\rangle = \frac{\int_\Gamma \tilde{\mu}(dx, \alpha) E_x^\alpha \int_0^\tau f(x(s))ds}{\int_\Gamma \tilde{\mu}(dx, \alpha) E_x^\alpha \tau}. \tag{3.3}$$

Equation (3.3) and various approximations to it will be widely used in the sequel.

Properties of $\{\tilde{X}_n\}$. The chain $\{\tilde{X}_n\}$ on state space Γ is said to be uniformly ϕ-recurrent (for a given measure ϕ on the Borel sets of Γ) if for each Borel $B \in \Gamma$ with $\phi(B) > 0$

$$P_x^\alpha\{\tilde{X}_i \in B, \text{ some } i \leq m\} \to 1 \quad \text{as } m \to \infty;$$

uniformly in $x \in \Gamma$. A sufficient condition [4, p. 29] is that if $\phi(B) > 0$, $\exists n < \infty$, $\varepsilon > 0$ (which can be B-dependent) such that

$$P_x^\alpha\{\tilde{X}_i \in B, \text{ some } i \leq n\} \geq \varepsilon, \text{ all } x \in \Gamma. \tag{3.4}$$

If the chain is ϕ-recurrent and a-periodic then $\exists C < \infty$, $\gamma < 1$ such that for Borel sets B

$$|P_x^\alpha\{\tilde{X}_n \in B\} - \tilde{\mu}(B,\alpha)| \leq C\gamma^n, \tag{3.5}$$

and for bounded measurable $f(\cdot)$,

$$|E_x^\alpha f(\tilde{X}_n) - \tilde{f}^\alpha| \leq 2C\gamma^n\|f - \tilde{f}^\alpha\|, \tag{3.6}$$

where $\|f\| = \sup_x |f(x)|$ and $\tilde{f}^\alpha = \langle\tilde{\mu}(\alpha), f\rangle$.

The next theorem follows from [3, p. 339, proof of Theorem 5.1 there]. The model in the reference does not explicitly include a parameter α, but it is easily seen from the proof of the cited theorem that the non-degeneracy and the fact that the moment bounds in Theorem 2.1 do not depend on $\alpha \in A_0$ implies that (3.4) is uniform in $\alpha \in A_0$ for some $\varepsilon > 0$. In fact, we can use $n = 1$. Actually, we will only need the result for $\alpha = \alpha_0$.

Theorem 3.1. *Assume* (A2.1)-(A2.3). $\{\tilde{X}_n\}$ *is ϕ-recurrent, where ϕ is Lebesgue measure on Γ. The recurrence is uniform in $\alpha \in A_0$ in the sense that the mean recurrence times are bounded uniformly for $\alpha \in A_0$. There are $C < \infty$, $\gamma < 1$ (not depending on $\alpha \in A_0$) such that* (3.5) *and* (3.6) *hold.*

It will be seen below (Lemma 4.1) that $\tilde{P}(x, n, B \mid \alpha)$ is continuous in x, uniformly in α, B. (The continuity is proved in the above reference [3], but we give a different proof since the details to be used will be needed elsewhere in the paper.)

4. The α-Derivative of $\tilde{\mu}(\alpha)$ (Setwise sense)

Let $C(\Gamma)$ denote the set of bounded and continuous functions on Γ, and $C_c(\Gamma)$ the centered functions: $f_1 \in C_c(\Gamma)$ if $f_1 = f - \tilde{f}$ for $f \in C(\Gamma)$, where $\tilde{f} = < \tilde{\mu}(\alpha_0), f >$. In order to prove the differentiability of $\mu(\alpha)$ at α_0, we first prove that of $\tilde{\mu}(\alpha)$, and then use (3.3).

Definition. $\tilde{\mu}(\alpha)$ is said to be *differentiable at* α_0 *in the setwise* (*or weak*) *sense* if there is a finite signed measure v such that for each Borel set B

$$v(B) = \lim_{\delta\alpha \to 0} \left[\tilde{\mu}(B, \alpha_0 + \delta\alpha) - \tilde{\mu}(B, \alpha_0)\right] / \delta\alpha.$$

$\tilde{\mu}(\alpha)$ is said to be *differentiable at* α_0 *in the sense of weak convergence* (or *weak** *sense*) if there is a finite signed measure v such that for each $f \in C(\Gamma)$,

$$\langle v, f \rangle = \lim_{\delta\alpha \to 0} \langle \tilde{\mu}(\alpha_0 + \delta\alpha) - \tilde{\mu}(\alpha_0), f \rangle / \delta\alpha.$$

Definition. Let $L^\infty(\Gamma)$ denote the bounded Borel measurable functions on Γ. For any Borel set H, let $\mathcal{B}(H)$ denote the Borel subsets of H. Define the operator $\tilde{P}(\alpha)$ on $L^\infty(\Gamma)$ by $\tilde{P}(\alpha) f(x) = E_x^\alpha f(\tilde{X}_1)$.

Lemma 4.1. *Assume* (A2.1)–(A2.4). *Then the set* $\{\tilde{P}(\alpha) L^\infty(\Gamma),\ \alpha \in A\}$ (*restricted to functions with* $\|f\| \leq 1$) *is equicontinuous.*

Proof. Define the process $y(\cdot)$ by $y(0) = x$ and

$$dy = \sigma(y) dw. \tag{4.1}$$

Define

$$\xi_0^\alpha(u, v) = \int_u^v \left[\sigma^{-1}(y(s)) b(y(s), \alpha)\right]' dw(s) - \frac{1}{2} \int_u^v |\sigma^{-1}(y(s)) b(y(s), \alpha)|^2 ds.$$

Given $\varepsilon > 0$, there are $T_2 > T_1 > T_0 > 0$ such that for all $\alpha \in A_0$ and $x \in \Gamma$,

$$P_x^\alpha\{\tau \geq T_2\} \leq \varepsilon, \quad P_x^\alpha\{\tau \leq T_1\} \leq \varepsilon \tag{4.2}$$

$$E_x^\alpha \exp \xi_0^\alpha(T_0, T_1) = 1. \tag{4.3}$$

$$\sup_{x \in \Gamma, \alpha \in A_0} E_x^\alpha \exp 2\xi_0^\alpha(0, T_1) \leq K_1^2 < \infty, \tag{4.4}$$

$$E^{1/2} |\exp \xi_0^\alpha(0, T_0) - 1|^2 \leq \varepsilon. \tag{4.5}$$

Let $\tau_{12} = (\tau \wedge T_2) \vee T_1$. By (4.2), we have

$$|E_x^\alpha f(\tilde{X}_1) - E_x^\alpha f(x(\tau_{12})| \leq 4\varepsilon \|f\|.$$

Write

$$E_x^\alpha f(x(\tau_{12})) = E_x^\alpha E_{x(T_1)}^\alpha f(x(\tau_{12})) = E_x^\alpha f_1(x(T_1)),$$

where f_1 is defined in the obvious way and $\|f_1\| \leq \|f\|$. By use of a Girsanov measure transformation, (4.4), (4.5) and Schwarz's inequality, we can write

$$\begin{aligned} E_x^\alpha f_1(x(T_1)) &= E_x^\alpha f_1(y(T_1)) \exp \xi_0^\alpha(0, T_1) \\ &= E_x^\alpha E_{y(T_0)}^\alpha f_1(y(T_1)) \exp \xi_0^\alpha(T_0, T_1) + \varepsilon' \\ &= E_x^\alpha f_2(y(T_0)) + \varepsilon', \end{aligned}$$

where $\|\varepsilon'\| \leq \varepsilon K_1 \|f\|$, f_2 is defined in the obvious way and $\|f_2\| \leq \|f\|$. Note that f_2 depends on α but $y(T_0)$ does not.

By the above estimates and arbitrariness of ε, we need only show the equicontinuity of the set $\{E_x^\alpha f_2(y(T_0)) : \|f_2\| \leq 1,\ \alpha \in A_0,\ f_2 \in L^\infty(\Gamma)\}$. Since $y(T_0)$ has a bounded density with respect to Lebesgue measure, using characteristic functions, we can write

$$E_x^\alpha f_2(y(T_0)) = \frac{1}{(2\pi)^r} \int f_2(y) dy \left\{ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (\exp -iu'y) E_x^\alpha \exp iu'y(T_0) du \right\}.$$

We have

$$|E_x^\alpha \exp iu'y(T_0)| \leq \exp -O(|u|^2),$$

where $O(\cdot)$ can be chosen independently of $x \in \Gamma$ and $\alpha \in A_0$. Also, the bracketed term is the density (modulo a proportionality factor $(\frac{1}{(2\pi)^r})$) of $y(T_0)$ and is bounded by $\exp -O(|y|^2)$, where $O(\cdot)$ can be chosen independently of $x \in \Gamma$ and $\alpha \in A_0$. Thus, we need only prove that

$$E_x^\alpha \exp iu'y(T_0)$$

is x-continuous on each bounded u-set. But this follows from the Feller property of $y(\cdot)$. Q.E.D.

Corollary. *Assume* (A2.1)–(A2.4). *Then the transition function* $\tilde{P}(x, n, B \mid \alpha) \equiv E_x^\alpha I_B(\tilde{X}_n)$ *is continuous in* x, *uniformly in* B, n *and* $\alpha \in A_0$. *Also* $\tilde{\mu}(B, \alpha_0 + \delta\alpha) \to \tilde{\mu}(B, \alpha_0)$, *uniformly in* $B \in \mathcal{B}(\Gamma)$.

Proof. The first assertion is a direct consequence of the lemma. Let $g \in L^\infty(\Gamma)$, $\|g\| \leq 1$. Then, by the invariance of $\tilde{\mu}(\alpha)$,

$$\langle \tilde{\mu}(\alpha_0 + \delta\alpha), g \rangle = \int \tilde{\mu}(dx, \alpha_0 + \delta\alpha) E_x^{\alpha_0 + \delta\alpha} g(\tilde{X}_1).$$

A measure transformation argument and the continuity of $b(\cdot)$ can be used to show that, as $\delta\alpha \to 0$, $E_x^{\alpha_0+\delta\alpha} g(\tilde{X}_1)$ converges to $E_x^{\alpha_0} g(\tilde{X}_1)$, uniformly in $x \in \Gamma$. The latter function is continuous on Γ by the lemma. In fact the continuity and the convergence is uniform in g. From this, the invariance of $\tilde{\mu}(\alpha_0)$ and the weak convergence $\tilde{\mu}(\alpha_0 + \delta\alpha) \Rightarrow \tilde{\mu}(\alpha_0)$ (see Lemma 4.3 below), we have

$$\begin{aligned} \lim_{\delta\alpha \to 0} \langle \tilde{\mu}(\alpha_0 + \delta\alpha), g \rangle &= \int \tilde{\mu}(dx, \alpha_0) E_x^{\alpha_0} g(\tilde{X}_1) \\ &= \int \tilde{\mu}(dx, \alpha_0) g(x), \end{aligned}$$

where the convergence is uniform in $g: \|g\| \leq 1$.   Q.E.D.

The next lemma will be used to get the differentiability of $\mu(\alpha)$ at α_0 from that of $\tilde{\mu}(\alpha)$, via (3.3).

Lemma 4.2. *Assume* (A2.1)–(A2.5). *Then for* $f \in L^\infty(\Gamma)$, *as* $\delta\alpha \to 0$

$$[\tilde{P}(\alpha_0 + \delta\alpha) - \tilde{P}(\alpha_0)]f/\delta\alpha$$

converges (uniformly in x) to the function with values $E_x^{\alpha_0} f(\tilde{X}_1)Z(\tau, \alpha_0)$. *The limit is continuous and the convergence is uniform for* $f: \|f\| \leq 1$. *The set*

$$\left\{E_x^{\alpha_0} Z(\tau, \alpha_0) f(\tilde{X}_1), \|f\| \leq 1, f \in L^\infty(\Gamma), \alpha \in A_0\right\}$$

is equicontinuous. The same result holds for the convergence

$$\frac{1}{\delta\alpha}\left[E_x^{\alpha_0+\delta\alpha} \int_0^\tau f(x(s))ds - E_x^{\alpha_0} \int_0^\tau f(x(s))ds\right]$$
$$\to E_x^{\alpha_0} \int_0^\tau f(x(s))ds\, Z(\tau, \alpha_0) = E_x^{\alpha_0} \int_0^\tau f(x(s))Z(s, \alpha_0)ds.$$

Proof. The proof of the last assertion is very similar to that of the prior assertions and will be omitted. By an argument analogous to that of Lemma 4.1, we can prove the equicontinuity of the cited set of functions. We will prove only the first assertion of the lemma. For $T < \infty$, via a Girsanov measure transformation,

$$\frac{E_x^{\alpha_0+\delta\alpha} f(x(\tau \wedge T)) - E_x^{\alpha_0} f(x(\tau \wedge T))}{\delta\alpha} = E_x^{\alpha_0} f(x(\tau \wedge T))[\exp \xi(0, T; \alpha_0, \delta\alpha) - 1]/\delta\alpha$$

$$= E_x^{\alpha_0} f(x(\tau \wedge T))[\exp \xi(0, \tau \wedge T; \alpha_0, \delta\alpha) - 1]/\delta\alpha. \tag{4.6}$$

We have, by (A2.5) and Theorem 2.1,

$$\lim_{\delta\alpha \to 0} \lim_{T \to \infty} E_x^{\alpha_0} \left[\frac{\exp \xi(0, \tau \wedge T; \alpha_0, \delta\alpha) - 1}{\delta\alpha} - Z(\tau, \alpha_0)\right]^2 = 0,$$

where the limit is attained uniformly in $x \in \Gamma$. The first assertion of the lemma follows from this and (4.6). Q.E.D.

The next corollary shows that the setwise derivative of $\tilde{\mu}(\alpha)$ at α_0 is absolutely continuous with respect to $\tilde{\mu}(\alpha_0)$.

Corollary. *Assume* (A2.1)–(A2.4). *Define the set function* $\tilde{v}$ *by*

$$\tilde{v}(B) = \lim_{\delta\alpha \to 0} \frac{1}{\delta\alpha} \int_\Gamma \tilde{\mu}(dx, \alpha_0)\big[\tilde{P}(x, 1, B \mid \alpha_0 + \delta\alpha) - \tilde{P}(x, 1, B \mid \alpha_0)\big].$$

Then there is $G \in L^1(\tilde{\mu}(\alpha_0))$ *such that* $\langle \tilde{\mu}(\alpha_0), G\rangle = 0$ *and*

$$\tilde{v}(B) = \int_B \tilde{\mu}(dx, \alpha_0) G(x).$$

The limit is uniform in B.

Proof. By the lemma, the limit is

$$\int_\Gamma \tilde{\mu}(dx, \alpha_0) E_x^{\alpha_0} Z(\tau, \alpha_0) I_B(\tilde{X}_1),$$

and the limit is taken on uniformly in B. (In fact $E_x^{\alpha_0} Z(\tau, \alpha_0) I_B(\tilde{X}_1)$ is continuous, uniformly in B.) Both $\tilde{\mu}(\alpha_0)$ and the measure defined by the limit are mutually absolutely continuous with respect to Lebesgue measure, since the transition probability $\tilde{P}(x, 1, \cdot \mid \alpha_0)$ is. Let G denote the Radon–Nikodym derivative of $\tilde{v}$ with respect to $\tilde{\mu}(\alpha_0)$. Since $E_x^{\alpha_0} Z(\tau, \alpha_0) I_{R^r}(\tilde{X}_1) = 0$, we have $\langle \tilde{\mu}(\alpha_0), G\rangle = 0$. Q.E.D.

Lemma 4.3. *Assume* (A2.1)–(A2.4). *Then* $\tilde{\mu}(\alpha_0 + \delta\alpha) \Rightarrow \tilde{\mu}(\alpha_0)$.

Proof. The proof follows from the uniqueness of $\tilde{\mu}(\alpha_0)$ and the convergence $\tilde{P}(x, 1, B \mid \alpha_0 + \delta\alpha) \to \tilde{P}(x, 1, B \mid \alpha_0)$, uniformly for $x \in \Gamma$ (Lemma 4.1), and the details are omitted.

Definition. Let $L_c^\infty(\Gamma) \subset L^\infty(\Gamma)$ be the 'centered' subset for which $\langle \tilde{\mu}(\alpha_0), f \rangle = 0$. We identify functions in $L_c^\infty(\Gamma)$ which are equal a.e. (Lebesgue measure).

The following lemma is a key result for proving the differentiability of $\tilde{\mu}(\alpha)$ at α_0. The representations used occur throughout the sequel.

Lemma 4.4. *Assume* (A2.1)-(A2.4). *Then* $(I - \tilde{P}(\alpha_0)) \colon L_c^\infty(\Gamma) \to L_c^\infty(\Gamma)$ *is invertible.*

Proof. The fact that $\tilde{P}(\alpha_0)$ maps $L_c^\infty(\Gamma)$ into $L_c^\infty(\Gamma)$ follows from the fact that $\tilde{\mu}(\alpha_0)$ is an invariant measure for the transition function $\tilde{P}(x, n, \cdot \mid \alpha_0)$. We prove the invertability by simply exhibiting the inverse. Let $f \in L_c^\infty(\Gamma)$. Then it is easily seen from (3.6) and the definition of $(I - \tilde{P}(\alpha_0))$ that the "inverse" defined by

$$(I - \tilde{P}(\alpha_0))^{-1} f(x) \equiv \sum_{n=0}^{\infty} \tilde{P}^n(\alpha_0) f(x) = \sum_{n=0}^{\infty} E_x^{\alpha_0} f(\tilde{X}_n) \tag{4.7}$$

satisfies our needs. Q.E.D.

Corollary. *Assume* (A2.1)-(A2.4). *Then* $(I - \tilde{P}(\alpha_0)) \colon C_c(\Gamma) \to C_c(\Gamma)$ *is invertable.*

Proof. By Lemma 4.1, $\tilde{P}(\alpha_0) C_c(\Gamma) \subset C_c(\Gamma)$. The rest of the proof is as for the lemma. Q.E.D.

Theorem 4.1. *Assume* (A2.1)-(A2.5). *Then* $\tilde{\mu}_\alpha(\alpha_0)$ *exists in the sense of setwise convergence and satisfies, for* $f \in L^\infty(\Gamma)$,

$$\langle \tilde{\mu}_\alpha(\alpha_0), f \rangle = \langle \tilde{\mu}(\alpha_0), \tilde{P}_\alpha^n(\alpha_0) f \rangle + \langle \tilde{\mu}_\alpha(\alpha_0), \tilde{P}^n(\alpha_0) f \rangle, \tag{4.8}$$

where

$$\tilde{P}_\alpha^n(\alpha_0) f(x) = \frac{d}{d\alpha} E_x^\alpha f(\tilde{X}_n) \Big|_{\alpha_0}.$$

Proof. For $f \in L_c^\infty(\Gamma)$, we have

$$\langle \tilde{\mu}(\alpha) - \tilde{\mu}(\alpha_0), f\rangle = \langle \tilde{\mu}(\alpha), \tilde{P}(\alpha) f\rangle - \langle \tilde{\mu}(\alpha_0), \tilde{P}(\alpha_0) f\rangle$$

$$\begin{aligned} (4.9) \qquad &= \langle \tilde{\mu}(\alpha) - \tilde{\mu}(\alpha_0), \tilde{P}(\alpha_0) f\rangle + \langle \tilde{\mu}(\alpha_0), (\tilde{P}(\alpha) - \tilde{P}(\alpha_0)) f\rangle \\ &+ \langle \tilde{\mu}(\alpha) - \tilde{\mu}(\alpha_0), (\tilde{P}(\alpha) - \tilde{P}(\alpha_0)) f\rangle. \end{aligned}$$

Write $\delta\tilde{\mu}(\alpha) = \tilde{\mu}(\alpha) - \tilde{\mu}(\alpha_0)$ and $\delta\tilde{P}(\alpha) = \tilde{P}(\alpha) - \tilde{P}(\alpha_0)$. Then, (4.9) yields

$$(4.10) \qquad \langle \delta\tilde{\mu}(\alpha)/\delta\alpha, (I - \tilde{P}(\alpha_0)) f\rangle = \langle \tilde{\mu}(\alpha_0), \frac{\delta\tilde{P}(\alpha)}{\delta\alpha} f\rangle + \langle \delta\tilde{\mu}(\alpha), \frac{\delta\tilde{P}(\alpha)}{\delta\alpha} f\rangle.$$

By Lemma 4.2 and either Lemma 4.3 or the Corollary to Lemma 4.1, the second right-hand term in (4.10) goes to zero as $\delta\alpha \to 0$ (uniformly in $f: \|f\| \leq 1$).

For $g \in L_c^\infty(\Gamma)$, define (use Lemma 4.4), $f = (I - \tilde{P}(\alpha_0))^{-1} g$. By Lemmas 4.2 and 4.4

$$\frac{\delta\tilde{P}(\alpha)}{\delta\alpha}(I - \tilde{P}(\alpha_0))^{-1} g$$

converges (uniformly in x) to the function with values

$$E_x^{\alpha_0} f(\tilde{X}_1) Z(\tau, \alpha_0) = E_x^{\alpha_0}[Z(\tau, \alpha_0) \sum_{n=0}^{\infty} E_y^{\alpha_0} g(\tilde{X}_n)\Big|_{y=\tilde{X}_1}] \equiv \tilde{g}(x).$$

which is in $C_c(\Gamma)$. Hence

$$(4.11) \qquad \lim_{\delta\alpha \to 0} \langle \delta\tilde{\mu}(\alpha)/\delta\alpha, g\rangle = \langle \tilde{\mu}(\alpha_0), \tilde{g}\rangle.$$

Since $g \in L_c^\infty(\Gamma)$, and $L_c^\infty(\Gamma) = L^\infty(\Gamma)$ modulo constant functions, (4.11) gives the desired setwise convergence.

The formula (4.8) follows in a similar way. Q.E.D.

Corollary. *Assume* (A2.1)-(A2.5). *Then* $\tilde{\mu}_\alpha(\alpha_0)$ *exists in the sense of weak convergence.*

Remark. The corollary is obviously a special case of the theorem. But, it can be proved directly via the method of proof of the theorem, simply by replacing all $L_c^\infty(\Gamma)$ by $C_c(\Gamma)$. This remark will be useful when working with the approximations in Section 7, since there we will have to work with weak convergence only.

Now that the existence of $\tilde{\mu}_\alpha(\alpha_0)$ is established, we can turn our attention to $\mu_\alpha(\alpha_0)$.

Theorem 4.2. *Assume* (A2.1)–(A2.5). *Then* $\mu_\alpha(\alpha_0)$ *exists in the sense of setwise convergence, and for* $f \in L^\infty(R^r)$,

$$\langle \mu_\alpha(\alpha_0), f\rangle = \frac{1}{\tilde{\mu}(R^r,\alpha_0)}\left[\int_\Gamma \tilde{\mu}(dx,\alpha_0)E_x^{\alpha_0}\int_0^\tau f(x(s))Z(s,\alpha_0)ds\right.$$
$$\left. + \int_\Gamma \tilde{\mu}_\alpha(dx,\alpha_0)E_x^{\alpha_0}\int_0^\tau f(x(s))ds\right]$$
$$-\frac{\langle \tilde{\mu}(\alpha_0), f\rangle}{(\tilde{\mu}(R^r,\alpha_0))^2}\left[\int_\Gamma \tilde{\mu}(dx,\alpha_0)E_x^{\alpha_0}\int_0^\tau Z(s,\alpha_0)ds + \int_\Gamma \tilde{\mu}_\alpha(dx,\alpha_0)E_0^\alpha \tau\right]$$

$$= \frac{d}{d\alpha}\left[\frac{\int \tilde{\mu}(dx,\alpha)E_x^\alpha \int_0^\tau f(x(s))ds}{\int \tilde{\mu}(dx,\alpha)E_x^\alpha \tau}\right]. \tag{4.12}$$

Also $\mu_\alpha(\alpha_0)$ *is absolutely continuous with respect to Lebesgue measure and has finite variation.*

Proof. Let $f \in L^\infty(R^r)$. Define $\delta\mu(\alpha) = \mu(\alpha_0+\delta\alpha) - \mu(\alpha_0)$ and define $\delta\tilde{\mu}(\alpha)$ analogously. Define the operator $\tilde{P}(\alpha)$ on $L^\infty(R^r)$ by $\tilde{P}(\alpha)f = E_x^\alpha \int_0^\tau f(x(s))ds$. Let e denote the function which is identically unity. We need to show the differentiability of

$$\langle \tilde{\mu}(\alpha), \tilde{P}(\alpha)f\rangle / \langle \tilde{\mu}(\alpha), \tilde{P}(\alpha)\mathrm{e}\rangle = \langle \mu(\alpha), f\rangle.$$

It will be sufficient to show the differentiability of the numerator only. This will be the first bracketed term in (4.12). We can write

$$\frac{1}{\delta\alpha}\left[\langle\tilde{\mu}(\alpha_0+\delta\alpha),\hat{P}(\alpha_0+\delta\alpha)f\rangle-\langle\tilde{\mu}(\alpha_0),\hat{P}(\alpha_0)f\rangle\right]$$

$$=\langle\frac{\delta\tilde{\mu}(\alpha)}{\delta\alpha},\hat{P}(\alpha_0)f\rangle+\langle\tilde{\mu}(\alpha_0),\frac{(\hat{P}(\alpha_0+\delta\alpha)-\hat{P}(\alpha_0))}{\delta\alpha}f\rangle$$

$$+\langle\delta\tilde{\mu}(\alpha_0),\frac{(\hat{P}(\alpha_0+\delta\alpha)-\hat{P}(\alpha_0))}{\delta\alpha}f\rangle.$$

By Lemma 4.2, the second term on the right converges to the first term in the first bracket on the right-hand side of (4.12). The first term on the right converges to the second term in the first bracket on the right-hand side of (4.12) by Theorem 4.1 and the fact that $\hat{P}(\alpha_0)f\in C(\Gamma)$. Similarly, the last term on the right goes to zero. The representation (4.12) implies the absolute continuity assertion since it equals zero if $f=0$ a.e. (Lebesgue measure). It also implies the finite variation. Q.E.D.

Theorem 4.3 essentially says that the α-derivative of $E_x^\alpha f(x(t))$ equals that of $\langle\mu(\alpha),f\rangle$ for large t.

Theorem 4.3. *Assume* (A2.1)–(A2.5). *Then for* $f\in L^\infty(R^r)$,

$$\lim_{t\to\infty}\frac{d}{d\alpha}\int\mu(dx,\alpha)E_x^\alpha f(x(t))\bigg|_{\alpha=\alpha_0}=\frac{d}{d\alpha}\langle\mu(\alpha),f\rangle\big|_{\alpha=\alpha_0}$$

$$=\lim_{t\to\infty}\int\mu(dx,\alpha_0)\left(E_x^{\alpha_0}f(x(t))\right)_\alpha,\tag{4.13}$$

$$\lim_{t\to\infty}\frac{d}{d\alpha}\int\mu(dx,\alpha)E_x^\alpha\frac{1}{t}\int_0^t f(x(s))ds\big|_{\alpha=\alpha_0}$$

$$=\lim_{t\to\infty}\int\mu(dx,\alpha_0)\left(E_x^{\alpha_0}f(x(t))\right)_\alpha,\tag{4.14}$$

and the limits exist.

Proof. By the differentiability proved in Theorem 4.2 we can write

$$\frac{d}{d\alpha}\int \mu(dx,\alpha)f(x) = \frac{d}{d\alpha}\int \mu(dx,\alpha)E_x^{\alpha}f(x(t))\big|_{\alpha=\alpha_0}$$

$$= \int \mu_\alpha(dx,\alpha_0)E_x^{\alpha_0}f(x(t)) + \int \mu(dx,\alpha_0)\big(E_x^{\alpha_0}f(x(t))\big)_\alpha. \tag{4.15}$$

As $t \to \infty$, $E_x^{\alpha_0}f(x(t)) \to \langle \mu(\alpha_0), f\rangle$ for $\mu(\alpha_0)$-almost all x. Since $\mu_\alpha(\alpha_0)$ is absolutely continuous with respect to Lebesgue measure (Theorem 4.2), and $\mu(\alpha_0)$ and Lebesgue measure are mutually absolutely continuous, we have that $\mu_\alpha(\alpha_0)$ is absolutely continuous with respect to $\mu(\alpha_0)$. Also $\langle \mu_\alpha(\alpha_0)$, constant function$\rangle = 0$. These facts imply that the first term on the right-hand side of (4.15) goes to zero as $t \to \infty$, which yields the assertion concerning (4.13). The expression (4.14) is proved in the same way. Q.E.D.

5. A Discrete Time Approximation

Since the paths of $x(\cdot)$ and $w(\cdot)$ are not physically available, we cannot evaluate (1.5) or use Theorem 4.2 or 4.3 as stated to get estimates of the derivatives $< \mu(\alpha_0), f >_\alpha$ via the use of paths of $x(\cdot)$ or $w(\cdot)$. We need to work with computable approximations to $x(\cdot)$ and $w(\cdot)$. In [1], two types of approximations were used for the finite time problem: the first was a discrete time approximation, and the second a Markov chain approximation. Each one has its own advantages, but simulation studies indicate that their overall numerical properties are similar. We will work with the discrete time approximation here. In this section, the approximation is defined. Some necessary stability results are proved in the next section. Among other things to be shown, the robustness properties of approximations to derivatives of invariant measures and ergodic costs will be clear.

For $\Delta > 0$ and $\delta w(n\Delta) = w(n\Delta + \Delta) - w(n\Delta)$, define $\{X_n^\Delta\}$ by $X_0^\Delta = x$ and

$$X_{n+1}^\Delta = X_n^\Delta + \Delta b(X_n^\Delta, \alpha_0) + \sigma(X_n^\Delta)\delta w(n\Delta). \tag{5.1}$$

Define the interpolation $x^\Delta(\cdot)$ to be the piecewise constant (on intervals $[n\Delta, n\Delta+\Delta)$) process with $x^\Delta(n\Delta) = X_n^\Delta$. Define $Z^\Delta(\cdot, \alpha_0)$ to be the piecewise constant (on intervals $[n\Delta, n\Delta + \Delta)$) process with value at $n\Delta$:

$$\begin{aligned} Z^\Delta(n\Delta, \alpha_0) &= \sum_{i=0}^{n-1} \left[\sigma^{-1}(X_i^\Delta, \alpha_0) b_\alpha(X_i^\Delta, \alpha_0)\right]' \delta w(i\Delta) \\ &= \sum_{i=0}^{n-1} \left[b_\alpha'(X_i^\Delta, \alpha_0) a^{-1}(X_n^\Delta)\right]\left[\delta X_i^\Delta - \Delta b(X_i^\Delta, \alpha_0)\right], \end{aligned}$$

where $\delta X_i^\Delta \equiv X_{i+1}^\Delta - X_i^\Delta$. For $T = N/\Delta$, [1, Section 4] shows that

$$Q^\Delta(\alpha_0) = \sum_{n=0}^{N-1} \Delta[k(X_n^\Delta, \alpha_0) Z^\Delta(n\Delta, \alpha_0) + k_\alpha(X_n^\Delta, \alpha_0)]$$

$$+ k_{0,\alpha}(X_N^\Delta, \alpha_0) + k_0(X_N^\Delta, \alpha_0) Z^\Delta(N\Delta, \alpha_0)$$

or the centered form

$$\hat{Q}^\Delta(\alpha_0) = \sum_{n=0}^{N-1} \Delta[(k(X_{n,\alpha_0}^\Delta) - E_x^{\alpha_0} k(X_n^\Delta, \alpha_0)) Z^\Delta(n\Delta, \alpha_0) + k_\alpha(X_n^\Delta, \alpha_0)]$$

$$+ k_{0,\alpha}(X_N^\Delta, \alpha_0) + (k_0(X_N^\Delta, \alpha_0) - \bar{k}_0^{\Delta, \alpha_0}) Z^\Delta(N\Delta, \alpha_0)$$

are appropriate approximations to (1.5). The $\hat{Q}^\Delta(\alpha_0)$ will have the smaller variance.

In fact we have

$$E_x^{\alpha_0} \hat{Q}^\Delta(\alpha_0) = E_x^{\alpha_0} Q^\Delta(\alpha_0) = \frac{d}{d\alpha} \left[E_x^\alpha \int_0^T k(x^\Delta(s), \alpha) ds + k_0(x^\Delta(T), \alpha) \right]_{\alpha=\alpha_0}$$

and we have the weak convergence

$$(Z^\Delta(\cdot, \alpha_0), Q^\Delta(\alpha_0), \hat{Q}^\Delta(\alpha_0), x^\Delta(\cdot)) \Rightarrow (Z(\cdot, \alpha_0), Q(\alpha_0), Q(\alpha_0), x(\cdot)).$$

We will obtain various 'infinite time' extensions of this result in Section 7.

Analogous to the comment below (1.5$'$), to reduce computation while exploiting the (variance reduction) advantages of the centering, in the simulations we replace $E_x^{\alpha_0} k(X_n^\Delta, \alpha_0)$ by $E_x^{\alpha_0} k(X_N^\Delta, \alpha_0)$, with good results in general.

6. Stability of The Approximation

An analog of Theorem 2.1 is needed for the $\{X_n^\Delta\}$ process. We will require the following additional condition

A6.1.

(a) $V_x'(x)b(x,\alpha) \to -\infty$ *as* $|x| \to \infty$, uniformly for $\alpha \in A_0$.

(b) $\liminf_{|x|\to\infty} \inf_{\alpha\in A_0} \dfrac{|V_x'(x)b(x,\alpha)|}{|b(x,\alpha)|^2} > 0.$

Theorem 6.1. *Assume* (A2.1)-(A2.3) *and* (A6.1). *There is a compact set Q such that for each compact $Q_1 \supset Q$, we have for small $\rho > 0$, $\delta > 0$, and $\Delta < \delta$,*

$$\sup_{\alpha\in A_0} \sup_{x\in Q_1-Q} E_x^\alpha \exp \rho\tau^\Delta < \infty, \tag{6.1}$$

where $\tau^\Delta = \min\{t: x^\Delta(t) \in Q\}$.

Proof. Let $X_0^\Delta = x$. For some $K_0 < \infty$, we have

$$\begin{aligned} A &\equiv E_x^\alpha \exp \rho[V(X_1^\Delta) - V(x)] \\ &\leq E_x^\alpha \exp \rho[V_x'(x)(b(x,\alpha)\Delta + \sigma(x)\delta w) + K_0(|\delta w|^2 + |b(x,\alpha)|^2\Delta^2)]. \end{aligned}$$

Note that for $2rk\Delta < 1$,

$$E_x^\alpha \exp k|\delta w|^2 \leq 1/(1-2rk\Delta).$$

Thus, for small ρ, Δ and $k_i > 0, 1/k_1 + 1/k_2 = 1$, Hölder's inequality yields

$$E_x^\alpha \exp \rho[V_x'(x)\sigma(x)\delta w + K_0|\delta w|^2]$$

$$\leq [\exp k_1^2\rho^2\Delta V_x'(x)a(x)V_x(x)/2]^{1/k_1} \cdot \frac{1}{(1-2r\rho k_2K_0\Delta)^{1/k_2}}.$$

Thus, for small ρ, Δ and k_1 fixed near unity,

$$A \leq \exp \rho[V_x'(x)b(x,\alpha)\Delta + K_0|b(x,\alpha)|^2\Delta^2$$

$$+\frac{k_1}{2}\rho\Delta V_x'(x)a(x)V_x(x) + 4rK_0\Delta].$$

Thus, there is a compact set Q and $\varepsilon_1 > 0$ such that for small ρ and for $x \notin Q$, $A \leq \exp -2\rho\varepsilon_1\Delta$. Thus for small ρ and $x \notin Q$,

$$E_x^\alpha \exp \rho\Delta\varepsilon_1 \cdot \exp \rho V(X_1^\Delta) \leq \exp \rho V(x).$$

Hence

$$E_x^\alpha \exp \rho\varepsilon_1\tau^\Delta \cdot \exp \rho V(x^\Delta(\tau^\Delta)) \leq \exp \rho V(x),$$

which yields the result, as in Theorem 2.1. Q.E.D.

7. Ergodic Properties of $\{X_n^\Delta\}$

We now set up the machinery so that results analogous to those in Sections 4 and 5 and the limits as $\Delta \to 0$ can be obtained. Define Γ, G, Γ_1 and G_1 as in Section 3. Define the stopping times:

$$\tau^{\Delta'} = \inf\{t : x^\Delta(t) \notin G_1 - \Gamma_1\},$$

$$\tau_1^\Delta = \inf\{t : x^\Delta(t) \in G\},$$

$$\tau_1^{\Delta'} = \inf\{t > \tau_1^\Delta : x^\Delta(t) \notin G_1 - \Gamma_1\}.$$

For $n > 1$,

$$\tau_n^\Delta = \inf\{t > \tau_{n-1}^{\Delta'} : x^\Delta(t) \in G\}$$

$$\tau_n^{\Delta'} = \inf\{t > \tau_n^\Delta : x^\Delta(t) \notin G_1 - \Gamma_1\}.$$

For $x = x^\Delta(0) \in G$, we use τ^Δ to denote $\tau_2^\Delta - \tau_1^\Delta = \tau_2^\Delta$, the canonical return time to G.

By Theorem 6.1, there are G, G_1 such that (e.g., let G equal the set Q of Theorem 6.1)

$$\sup_{x\in G, \alpha\in A_0} E_x^\alpha \tau^\Delta < \infty, \quad \sup_{x\in G,\alpha\in A_0} E_x^\alpha \exp \rho\tau^\Delta < \infty, \tag{7.1}$$

for small ρ. Define $\tilde{X}_n^\Delta = x^\Delta(\tau_n^\Delta)$. For $\alpha \in A_0$, the process $\{\tilde{X}_n^\Delta,\ n \geq 0\}$ is a homogeneous positive recurrent Markov chain with state space G. Let $\tilde{P}^\Delta(x, n, \cdot \mid \alpha)$ denote the transition function. There is a unique invariant measure $\tilde{\mu}^\Delta(\alpha)$. Analogously to the situation in Section 3, define the following:

$$\tau^\Delta(A) = \int_0^{\tau^\Delta} I_A(x^\Delta(s))ds, \quad A = \text{Borel set in } R^r,$$

$$\bar{\mu}^\Delta(A, \alpha) = \int_G \tilde{\mu}^\Delta(dx, \alpha) E_x^\alpha \tau^\Delta(A)$$

$$\mu^\Delta(A, \alpha) = \bar{\mu}^\Delta(A, \alpha)/\bar{\mu}^\Delta(R^r, \alpha).$$

The same argument used to show that $\mu(\alpha)$ is invariant for $x(\cdot)$ ([2], p. 183) can be used to show that $\mu^{\Delta}(\alpha)$ is invariant for $\{X_n^{\Delta}\}$, under parameter α. We can now write for bounded measurable f:

$$(7.2) \qquad \langle \mu^{\Delta}(\alpha), f\rangle = \frac{\int_G \tilde{\mu}^{\Delta}(dx,\alpha)E_x^{\alpha}\int_0^{\tau^{\Delta}} f(x^{\Delta}(s))ds}{\int_G \tilde{\mu}^{\Delta}(dx,\alpha)E_x^{\alpha}\tau^{\Delta}}.$$

Let $L^{\infty}(G)$ denote the set of bounded Borel measurable functions on G. Define the operator $\tilde{P}^{\Delta}(\alpha)$ on $L^{\infty}(G)$ by $\tilde{P}^{\Delta}(\alpha)f(x) = E_x^{\alpha} f(\tilde{X}_1^{\Delta})$, $x \in G$.

Lemma 7.1. *Assume* (A2.1)-(A2.4) *and* (A6.1). *Then the set* $\{\tilde{P}^{\Delta}(\alpha)L^{\infty}(G)$ (*restricted to* $\|f\| \le 1$), $\Delta > 0$, $\alpha \in A_0\}$ *is equicontinuous.*

Remark on the proof. Define the process $\bar{y}^{\Delta}(\cdot)$ to be the piecewise constant interpolation (intervals $[n\Delta, n\Delta+\Delta)$) of the process defined by $\bar{Y}_0^{\Delta} = x$, $\bar{Y}_{n+1}^{\Delta} = \bar{Y}_n^{\Delta} + \sigma(\bar{Y}_n^{\Delta})\delta w(n\Delta)$. Then $\bar{y}^{\Delta}(\cdot) \Rightarrow y(\cdot)$, defined in Lemma 4.1. Define the Radon-Nikodym derivative $\exp \xi_0^{\alpha,\Delta}(0,T)$, where

$$\begin{aligned}\xi_0^{\alpha,\Delta}(0,T) = &\sum_{n=0}^{T/\Delta-1} [\sigma^{-1}(\bar{Y}_n^{\Delta})b(\bar{Y}_n^{\Delta},\alpha)]'\delta w(n\Delta) \\ &- \frac{1}{2}\sum_{n=0}^{T/\Delta-1} |\sigma^{-1}(\bar{Y}_n^{\Delta})b(\bar{Y}_n^{\Delta},\alpha)|^2\Delta.\end{aligned}$$

From this point on, the proof is nearly identical to that of Lemma 4.1 and is omitted.

Theorem 7.1. *Assume* (A2.1)-(A2.4) *and* (A6.1) *and let* $\alpha = \alpha_0$. *Then* $\tilde{X}_k^{\Delta} \Rightarrow \tilde{X}_k$ *if* $\tilde{X}_0^{\Delta} \Rightarrow \tilde{X}_0$, *and* $\tilde{\mu}^{\Delta}(\alpha_0) \Rightarrow \tilde{\mu}(\alpha_0)$. *In addition* $E_x^{\alpha_0} f(\tilde{X}_k^{\Delta}) \xrightarrow{\Delta} E_x^{\alpha_0} f(\tilde{X}_k)$ *uniformly in* $x \in G$ *and in* f *in any equicontinuous set with* $\|f\| \le 1$. *Also,* $\tilde{\mu}^{\Delta}(\alpha_0 + \delta\alpha) \Rightarrow \tilde{\mu}^{\Delta}(\alpha_0)$ *and* $\mu^{\Delta}(\alpha_0) \Rightarrow \mu(\alpha_0)$. *Finally,*

$$\bar{f}^{\Delta,\alpha_0} \equiv \langle \mu^{\Delta}(\alpha_0), f\rangle \to \langle \mu(\alpha_0), f\rangle \equiv \bar{f}^{\alpha_0},$$

uniformly for f in any equicontinuous set with $\|f\| \leq 1$.

Proof. Note that $(x^{\Delta}(\cdot), \tau^{\Delta}) \to (x(\cdot), \tau)$ uniformly in $x \in G$ in the sense that $E_x^{\alpha_0} F(x^{\Delta}(\cdot), \tau^{\Delta}) \to E_x^{\alpha_0} F(x(\cdot), \tau)$ uniformly in $x \in G$, for any bounded and continuous real valued $F(\cdot)$. The weak convergence $\tilde{X}_k^{\Delta} \Rightarrow \tilde{X}_k$ (if $\tilde{X}_0^{\Delta} \Rightarrow \tilde{X}_0$) follows from the uniform integrability of $\{\tau^{\Delta}, \Delta > 0, \alpha \in A_0\}$ and the (uniform) weak convergence of $x^{\Delta}(\cdot)$ to $x(\cdot)$. The asserted weak convergence can be proved by a standard martingale method [5], [6] (and using the non-degeneracy of $a(\cdot)$ and the smoothness of Γ, Γ_1 to get the weak convergence of τ^{Δ}). In fact, a standard weak convergence method can be used to get $\tilde{P}^{\Delta}(\alpha_0)f \to \tilde{P}(\alpha_0)f$, uniformly in f in any equicontinuous set in $C(G)$.

Now, for $f \in C(G)$, by the invariance of $\mu^{\Delta}(\alpha_0)$, we can write

$$\langle \tilde{\mu}^{\Delta}(\alpha_0), f \rangle = \langle \tilde{\mu}^{\Delta}(\alpha_0), \tilde{P}^{\Delta}(\alpha_0) f \rangle \equiv \bar{f}^{\Delta, \alpha_0}.$$

$\{\tilde{\mu}^{\Delta}(\alpha_0), \Delta > 0\}$ is obviously tight since G is compact. If $\hat{\mu}(\alpha_0)$ is the limit of a weakly convergent subsequence, then by the last expression, we have

$$\langle \hat{\mu}(\alpha_0), f \rangle = \langle \hat{\mu}(\alpha_0), \tilde{P}(\alpha_0) f \rangle, \quad f \in C(G),$$

which yields $\hat{\mu}(\alpha_0) = \tilde{\mu}(\alpha_0)$.

Now use (7.2), the weak convergence $\{\tau^{\Delta}, x^{\Delta}(\cdot)\} \Rightarrow \{\tau, x(\cdot)\}$ and the uniform integrability of $\{\tau^{\Delta}\}$ (Theorem 6.1) and $\tilde{\mu}^{\Delta}(\alpha_0) \Rightarrow \tilde{\mu}(\alpha_0)$ to get $\mu^{\Delta}(\alpha_0) \Rightarrow \mu(\alpha_0)$. The last assertion of the theorem is also proved by an argument by contradiction and the proof is omitted. Q.E.D.

An analog of (3.6). The following lemma is needed to get an analog of Lemma 4.4.

Lemma 7.2. *Assume* (A2.1)–(A2.3) *and* (A6.1). *Let k be such that* $C\gamma^k \equiv$

$\lambda < 1$ (*see* (3.5)). *Let* $C'(G) \subset C(G)$ *be an equicontinuous set. Then*

$$\overline{\lim}_{\Delta\to 0} \sup_{f\in C'(G)} \frac{|E_x^{\alpha_0} f(\tilde{X}_k^{\Delta}) - \tilde{f}^{\Delta,\alpha_0}|}{\|f - \tilde{f}^{\Delta,\alpha_0}\|} \leq \lambda. \tag{7.3}$$

Equivalently, there are $\psi_1 < 1$, $C_1 < \infty$, *such that for small* $\Delta > 0$,

$$\|\tilde{P}^{\Delta}(\alpha_0)^n f - \tilde{f}^{\Delta,\alpha_0}\| \leq C_1 \psi_1^n \|f - \tilde{f}^{\Delta,\alpha_0}\|. \tag{7.4}$$

Proof. Suppose that (7.3) is false. Then there is $x_n \to x \in G$, $\Delta_n \to 0$, $\lambda_n \geq \lambda_0 > \lambda$, $f_n \in C'(G)$, $f_n \to f \in C'(G)$, such that

$$\frac{|E_{x_n}^{\alpha_0} f_n(\tilde{X}_k^{\Delta_n}) - \tilde{f}_n^{\Delta_n,\alpha_0}|}{\|f_n - \tilde{f}_n^{\Delta_n,\alpha_0}\|} \geq \lambda_n.$$

Without loss of generality, we can suppose that the infima of the denomenators are positive. Then we can write

$$\begin{aligned} \frac{|E_x^{\alpha_0} f(\tilde{X}_k) - \tilde{f}^{\alpha_0}|}{\|f_n - \tilde{f}_n^{\Delta_n,\alpha_0}\|} &\geq \frac{|E_{x_n}^{\alpha_0} f_n(\tilde{X}_k^{\Delta_n}) - \tilde{f}_n^{\Delta_n,\alpha_0}|}{\|f_n - \tilde{f}_n^{\Delta_n,\alpha_0}\|} \\ &- \frac{|E_x^{\alpha_0} f(\tilde{X}_k) - E_{x_n}^{\alpha_0} f_n(\tilde{X}_k^{\Delta_n})|}{\|f_n - \tilde{f}_n^{\Delta_n,\alpha_0}\|} - \frac{|f^{\alpha_0} - \tilde{f}_n^{\Delta_n,\alpha_0}|}{\|f_n - \tilde{f}_n^{\Delta_n,\alpha_0}\|}. \end{aligned} \tag{7.5}$$

The last two terms on the right go to zero by the weak convergence $\tilde{X}_k^{\Delta_n} \Rightarrow \tilde{X}_k$ (initial conditions $\tilde{X}_0^{\Delta} = x_n$, $\tilde{X}_0 = x$, resp.), and $\tilde{\mu}^{\Delta_n}(\alpha_0) \Rightarrow \tilde{\mu}(\alpha_0)$, and the convergence $f_n \to f$. The left side of (7.5) goes to $|E_x^{\alpha_0} f(\tilde{X}_k) - \tilde{f}^{\alpha_0}|/\|f - \tilde{f}^{\alpha_0}\| \leq C\psi^k = \lambda$ and we have a contradiction.

Inequality (7.4) follows from (7.3) by letting $\psi_1^k = (\lambda + \delta\lambda)$ for small $\delta\lambda > 0$, and iterating. Q.E.D.

Lemma 7.3. *Assume* (A2.1)-(A2.5) *and* (A6.1). *Then for* $f \in L^\infty(G)$,

$$\frac{[\tilde{P}^{\Delta}(\alpha_0 + \delta\alpha) - \tilde{P}^{\Delta}(\alpha_0)]f}{\delta\alpha}$$

converges (as $\delta\alpha \to 0$) to the function $\tilde{P}^{\Delta}_{\alpha}(\alpha_0)f$ with values

$$E^{\alpha_0}_x Z^{\Delta}(\tau^{\Delta}, \alpha_0) f(\tilde{X}^{\Delta}_1) = \frac{d}{d\alpha} E^{\alpha}_x f(\tilde{X}^{\Delta}_1)\big|_{\alpha=\alpha_0}.$$

The limit is continuous and the convergence is uniform in $\Delta, x \in G$, and in $f \in C(G)$ for $\|f\| \le 1$. The set $\{E^{\alpha_0}_x Z^{\Delta}(\tau^{\Delta}, \alpha_0) f(\tilde{X}^{\Delta}_1),\ \Delta > 0,\ f \in C(G),\ \|f\| \le 1\}$ is equicontinuous.

The same result holds for the convergence

$$\frac{1}{\delta\alpha}\left[E^{\alpha_0+\delta\alpha}_x \int_0^{\tau^{\Delta}} f(x^{\Delta}(s))ds - E^{\alpha_0}_x \int_0^{\tau^{\Delta}} f(x^{\Delta}(s))ds\right]$$

$$\to E^{\alpha_0}_x \int_0^{\tau^{\Delta}} f(x^{\Delta}(s)) Z^{\Delta}(s, \alpha_0) ds.$$

The proof is analogous to that of Lemma 4.2 but uses the Radon–Nikodym derivative introduced in the remark under Lemma 7.1, and is omitted.

Theorem 7.2. *Assume* (A2.1)–(A2.5) *and* (A6.1). *Then $\tilde{\mu}^{\Delta}_{\alpha}(\alpha_0)$ and $\mu^{\Delta}_{\alpha}(\alpha_0)$ exist in the sense of weak convergence.*

Proof. Let $C^{\Delta}_c(G)$ be the subset of $C(G)$ for which $\langle f, \tilde{\mu}^{\Delta}(\alpha_0)\rangle = 0$. Following the proof of Lemma 4.4 and its corollary, we first show the invertability of $(I - \tilde{P}^{\Delta}(\alpha_0))$ on $C_c(G)$, on which we identify functions which are equal a.e. $(\tilde{\mu}^{\Delta}(\alpha_0))$. By Lemma 7.1 and the fact that $\tilde{\mu}^{\Delta}(\alpha_0)$ is an invariant measure for the transition function which defines $\tilde{P}^{\Delta}(\alpha_0)$, for $f \in C^{\Delta}_c(G)$ the sum below converges and we have $(I - \tilde{P}^{\Delta}(\alpha_0))C^{\Delta}_c(G) \subset C^{\Delta}_c(G)$. By Lemma 7.2, we obviously have

$$(I - \tilde{P}^{\Delta}(\alpha_0)) \sum_{n=0}^{\infty} (\tilde{P}^{\Delta}(\alpha_0))^n f = \sum_{n=0}^{\infty} (\tilde{P}^{\Delta}(\alpha_0))^n (I - \tilde{P}^{\Delta}(\alpha_0)) f = f.$$

These facts yield that the inverse is

$$g^{\Delta} = (I - \tilde{P}^{\Delta}(\alpha_0))^{-1} f = \sum_{n=0}^{\infty} (\tilde{P}^{\Delta}(\alpha_0))^n f. \tag{7.6}$$

By Lemmas 7.1 and 7.2, the sum on the right side converges uniformly in Δ and it is equicontinuous for $f \in C_c^\Delta(G)$, $\|f\| \leq 1$, $\Delta > 0$.

We can now use a proof analogous to that of Theorem 4.1 (but using weak rather than setwise convergence) together with Lemma 7.3 and the weak convergence $\tilde{\mu}^\Delta(\alpha_0 + \delta\alpha) \Rightarrow \tilde{\mu}^\Delta(\alpha_0)$ to get the existence of $\tilde{\mu}_\alpha^\Delta(\alpha_0)$ in the sense of weak convergence, and the few details are omitted. To get the existence of $\mu_\alpha^\Delta(\alpha_0)$ in the sense of weak convergence, use the representation (7.2) and the α-differentiability of $\tilde{\mu}^\Delta(\alpha)$, $E_x^\alpha \int_0^{\tau^\Delta} f(x^\Delta(s))ds$ at $\alpha = \alpha_0$, and $E_x^{\alpha_0}\tau^\Delta > 0$. The details are like those of Theorem 4.2, but uses the equicontinuity of $\{E_x^\alpha \int_0^{\tau^{\alpha\Delta}} f(x^\Delta(s))ds\}$ (in $f \in C(G)$, $\Delta > 0$, $\|f\| \leq 1$, $\alpha \in A_0$), the weak convergence, and the uniform integrability of $\{\tau^\Delta$, small $\Delta > 0, \alpha \in A_0\}$. Q.E.D.

Corollary. *Assume the conditions of the Theorem. Then $\tilde{\mu}_\alpha^\Delta(\alpha_0)$ exists in the sense of setwise convergence. Also $\{\tilde{\mu}_\alpha^\Delta(\alpha_0)$, small $\Delta > 0\}$ is of bounded variation. For $g \in L^\infty(G)$, there is a unique $f \in L^\infty(G)$ such that*

$$(I - \tilde{P}^\Delta(\alpha_0))f = g - \tilde{g}^{\Delta,\alpha_0}$$

and $\tilde{f}^{\Delta,\alpha_0} = 0$.

Proof. Let $f \in L^\infty(G)$. Analogous to (4.9), write $\delta\tilde{P}(\alpha) = \tilde{P}^\Delta(\alpha_0 + \delta\alpha) - \tilde{P}^\Delta(\alpha_0), \delta\tilde{\mu}^\Delta(\alpha) = \tilde{\mu}^\Delta(\alpha_0 + \delta\alpha) - \tilde{\mu}^\Delta(\alpha_0)$, and

$$< \frac{\delta\tilde{\mu}^\Delta(\alpha)}{\delta\alpha}, f >=< \frac{\delta\tilde{\mu}^\Delta(\alpha)}{\delta\alpha}, \tilde{P}^\Delta(\alpha_0)f > \tag{7.7}$$

$$+ < \tilde{\mu}^\Delta(\alpha_0), \frac{\delta\tilde{P}^\Delta(\alpha)}{\delta\alpha}f > + < \delta\tilde{\mu}^\Delta(\alpha), \frac{\delta\tilde{P}^\Delta(\alpha_0)}{\delta\alpha}f > .$$

By Lemma 7.3, $(\delta\tilde{P}^\Delta(\alpha)/\delta\alpha)f$ converges to a continuous function, uniformly in $x \in G$. This and $\delta\tilde{\mu}^\Delta(\alpha) \Rightarrow$ zero measure implies that the last term on the right of (7.7) tends to zero, as $\delta\alpha \to 0$. Furthermore, since $\tilde{\mu}_\alpha^\Delta(\alpha_0)$ exists in the sense of weak convergence by the theorem. The second term on the right of (7.7) tends to $< \tilde{\mu}^\Delta(\alpha_0), \tilde{P}_\alpha^\Delta(\alpha_0) f >$.

Since $\tilde{P}^\Delta(\alpha_0)f$ is continuous (Lemma 7.1), and $\tilde{\mu}_\alpha^\Delta(\alpha_0)$ exists in the sense of weak convergence, the first term on the right tends to $< \tilde{\mu}_\alpha^\Delta(\alpha_0), \tilde{P}^\Delta(\alpha_0) f >$. Thus the limit of the left side of (7.7) exists. Now, the form of the limit of the right side implies that $\tilde{\mu}_\alpha^\Delta(\alpha_0)$ exists in the sense of setwise convergence.

Rewrite (7.7) as

$$< \tilde{\mu}_\alpha^\Delta(\alpha_0), (I - \tilde{P}^\Delta(\alpha_0)) f > = < \tilde{\mu}^\Delta(\alpha_0), \tilde{P}^\Delta(\alpha_0) f > .$$

For $g \in L^\infty(G)$, set $\tilde{g} = g - \bar{g}^{\Delta,\alpha_0}$ and define

$$f^\Delta = \sum_{n=0}^{\infty} (\tilde{P}^\Delta(\alpha_0))^n \tilde{g} \tag{7.8}$$

$$= \tilde{g} + \sum_{n=0}^{\infty} (\tilde{P}^\Delta(\alpha_0))^n (\tilde{P}^\Delta(\alpha_0)\tilde{g}).$$

The sum converges uniformly in g, Δ, for $\| g \| \leq 1$, since $\{\tilde{P}^\Delta(\alpha_0)g, \Delta > 0, g \in L^\infty(G), \|g\| \leq 1\}$ is equicontinuous by Lemma 7.1. The uniqueness assertion follows.

Thus

$$(I - \tilde{P}^\Delta(\alpha_0)) f^\Delta = \tilde{g}$$

and

$$< \tilde{\mu}_\alpha^\Delta(\alpha_0), \tilde{g} > = < \tilde{\mu}_\alpha^\Delta(\alpha_0), g > = < \tilde{\mu}^\Delta(\alpha_0), \tilde{P}^\Delta(\alpha_0) f^\Delta > .$$

The bounded variation assertion follows from this representation. Q.E.D.

The convergence Theorem for the discretizations.

Theorem 7.3. *Assume* (A2.1)–(A2.5) *and* (A6.1). *Then* $\tilde{\mu}_\alpha^\Delta(\alpha_0)$ *converges setwise to* $\tilde{\mu}_\alpha(\alpha_0)$ *and* $\mu_\alpha^\Delta(\alpha_0)$ *converges setwise to* $\mu_\alpha(\alpha_0)$.

Proof. Let $f \in L^\infty(G)$. Let g^Δ and g, resp., be the unique solutions in $L^\infty(G)$ (Theorem 7.2, Lemma 4.4) to

$$(I - \tilde{P}^\Delta(\alpha_0))g^\Delta = f - \tilde{f}^{\Delta,\alpha_0}$$

$$(I - \tilde{P}(\alpha_0))g = f - \tilde{f}^{\alpha_0}.$$

Note that

$$\frac{d}{d\alpha}\langle\tilde{\mu}^\Delta(\alpha), g^\Delta\rangle|_{\alpha=\alpha_0} = \langle\tilde{\mu}_\alpha^\Delta(\alpha_0), g^\Delta\rangle$$

$$= \langle\tilde{\mu}_\alpha^\Delta(\alpha_0), \tilde{P}^\Delta(\alpha_0)g^\Delta\rangle + \langle\tilde{\mu}^\Delta(\alpha_0), \tilde{P}_\alpha^\Delta(\alpha_0)g^\Delta\rangle.$$

Then we can write

$$\langle\tilde{\mu}_\alpha^\Delta(\alpha_0), f\rangle = \langle\tilde{\mu}_\alpha^\Delta(\alpha_0), f - \tilde{f}^{\Delta,\alpha_0}\rangle =$$

$$= \langle\tilde{\mu}_\alpha^\Delta(\alpha_0), (I - \tilde{P}^\Delta(\alpha_0))g^\Delta\rangle \tag{7.9}$$

$$= \int \tilde{\mu}^\Delta(dx, \alpha_0)\frac{d}{d\alpha}E_x^\alpha g^\Delta(\tilde{X}_1^\Delta)\Big|_{\alpha=\alpha_0}.$$

We have

$$\frac{d}{d\alpha}E_x^\alpha g^\Delta(\tilde{X}_1^\Delta)\Big|_{\alpha=\alpha_0} = E_x^{\alpha_0} g^\Delta(\tilde{X}_1^\Delta)Z^\Delta(\tau^\Delta, \alpha_0).$$

Now, note that the sum in (7.6) converges uniformly in Δ (Lemma 7.2); hence $g^\Delta \to g$, since $\tilde{P}^\Delta(\alpha_0)^n f \to \tilde{P}(\alpha_0)^n f$. Using this, the weak convergence of $\{\tilde{X}_1^\Delta, Z^\Delta(\tau^\Delta, \alpha_0)\}$, the uniform integrability of $\{Z^\Delta(\tau^\Delta, \alpha_0),\ \Delta > 0\}$, the fact that the functions on the right side of (7.9) converge uniformly in $x \in G$ to the continuous limit, and the fact that $\tilde{\mu}^\Delta(\alpha_0) \Rightarrow \tilde{\mu}(\alpha_0)$ yields that the limit as $\Delta \to 0$ of the right side of (7.9) is

$$\int \tilde{\mu}(dx, \alpha_0)E_x^{\alpha_0} g(\tilde{X}_1)Z(\tau, \alpha_0) = \int \tilde{\mu}(dx, \alpha_0)\frac{d}{d\alpha}E_x^\alpha g(\tilde{X}_1)\Big|_{\alpha=\alpha_0} =$$

$$= \langle \tilde{\mu}_\alpha(\alpha_0), (I - \tilde{P}(\alpha_0))g \rangle = \langle \tilde{\mu}_\alpha(\alpha_0), f - \tilde{f}^{\alpha_0} \rangle = \langle \tilde{\mu}_\alpha(\alpha_0), f \rangle.$$

Thus

$$\langle \tilde{\mu}_\alpha^\Delta(\alpha_0), f \rangle \to \langle \tilde{\mu}_\alpha(\alpha_0), f \rangle$$

which yields the setwise convergence of $\tilde{\mu}_\alpha^\Delta(\alpha_0)$ to $\tilde{\mu}_\alpha(\alpha_0)$. The setwise convergence of $\mu^\Delta(\alpha_0)$ to $\mu(\alpha_0)$ follows from the representation (7.2). For example to get the limit of the derivative of the denominator of (7.2), note that the derivative of the denominator is

$$\int_G \tilde{\mu}_\alpha^\Delta(dx, \alpha_0) E_x^{\alpha_0} \tau^\Delta + \int_G \tilde{\mu}_\alpha^\Delta(dx, \alpha_0) \frac{d}{d\alpha} E_x^\alpha \tau^\Delta \big|_{\alpha=\alpha_0}.$$

Then use the representation

$$\frac{d}{d\alpha} E_x^\alpha \tau^\Delta \bigg|_{\alpha=\alpha_0} = E_x^{\alpha_0} \tau^\Delta Z^\Delta(\tau^\Delta, \alpha_0),$$

and the proved convergence and uniform integrability (where appropriate) results for $\tilde{\mu}_\alpha^\Delta(\alpha_0)$, $X^\Delta(\cdot), \tau^\Delta, Z^\Delta(\tau^\Delta, \alpha_0)$. Q.E.D.

A finite time approximation Theorem. The next result shows that the derivative $\langle \mu_\alpha(\alpha_0), f \rangle$ of the ergodic cost can be arbitrarily well approximated by $\frac{d}{d\alpha} E_x^\alpha f(x^\Delta(t))\big|_{\alpha=\alpha_0}$ for large t and small Δ. It is such approximations that are actually used in the applications. It is important to note that for large enough t, the quality of the approximation is uniformly good in (small) Δ.

Theorem 7.4. *Assume* (A2.1)-(A2.5), (A6.1). *Then for* $f \in L^\infty(R^r)$,

$$\langle \mu_\alpha(\alpha_0), f \rangle = \lim_{\Delta \to 0} \langle \mu_\alpha^\Delta(\alpha_0), f \rangle$$

$$= \lim_{\substack{\Delta \to 0 \\ t \to \infty}} \int \mu^\Delta(dx, \alpha_0) \frac{d}{d\alpha} E_x^\alpha f(x^\Delta(t))\big|_{\alpha=\alpha_0}, \tag{7.10}$$

where the limits as $\Delta \to 0$, $t \to \infty$ *can be taken in any way at all.*

Proof. Write, by the invariance of $\mu^{\Delta}(\alpha)$ and the differentiability:

$$\frac{d}{d\alpha}\langle \mu^{\Delta}(\alpha), f\rangle_{\alpha=\alpha_0} = \langle \mu^{\Delta}_{\alpha}(\alpha_0), P^{\Delta}(\alpha_0,t)f\rangle$$

$$+ \langle \mu^{\Delta}(\alpha_0), P^{\Delta}_{\alpha}(\alpha_0,t)f\rangle, \tag{7.11}$$

where $P^{\Delta}(\alpha_0,t)f(x) = E^{\alpha_0}_x f(x^{\Delta}(t))$ and $t > 0$. We have $P^{\Delta}(\alpha_0,t)f \to \bar{f}^{\alpha_0} = < \mu(\alpha_0), f >$ as $\Delta \to 0$, $t \to \infty$. Also, $\{\mu^{\Delta}_{\alpha}(\alpha_0),\ \Delta > 0\}$ is of bounded variation by the corollary to Theorem 7.2. Thus $\langle \mu^{\Delta}_{\alpha}(\alpha_0), \tilde{P}^{\Delta}(\alpha_0,t)f\rangle \to 0$ as $\Delta \to 0$ and $t \to \infty$, which yields the theorem. Q.E.D.

A pathwise result. With the approximation of Theorem 7.4 in hand, we can give the pathwise result. Since we only have one long realization and cannot explicitly calculate the derivatives of the expectations, we need to show that a long simulation of $\{X^{\Delta}_n,\ n < \infty\}$ can yield a good approximation to the right side of (7.10) for fixed Δ. Typically, the t_0 in Theorem 7.5 is as large as can be, consistent with a modest sample variance.

Theorem 7.5. *Assume* (A2.1)-(A2.5) *and* (A6.1). *Fix* $t = n\Delta$. *Let* $f(\cdot)$ *be bounded and continuous. Then as* $T \to \infty$ *(or with centered* f *used as discussed in Section 5)*

$$\frac{1}{T}\int_0^T \left[Z^{\Delta}(t_0+s,\alpha_0) - Z^{\Delta}(s,\alpha_0)\right] f(x^{\Delta}(t_0+s))ds$$

$$\xrightarrow{P} \int \mu^{\Delta}(dx,\alpha_0) E^{\alpha_0}_x Z^{\Delta}(t_0,\alpha_0) f(x^{\Delta}(t_0)) \tag{7.12}$$

$$= \langle \mu^{\Delta}(\alpha_0), \frac{d}{d\alpha} E^{\alpha} f(x^{\Delta}(t_0))\big|_{\alpha=\alpha_0}\rangle.$$

Proof. Fix t_0. Define $\delta Z^{\Delta}(t_0,s) = Z^{\Delta}(t_0+s,\alpha_0) - Z^{\Delta}(s,\alpha_0)$ and $Y^{\Delta}(t_0,s) = \delta Z^{\Delta}(t_0,s) f(x^{\Delta}(t_0+s))$. Then the process (parameter T) defined by

$$M^{\Delta}(T) = \int_0^T \left[Y^{\Delta}(t_0,s) - E^{\alpha_0}_{x^{\Delta}(s)} Y^{\Delta}(t_0,s)\right] ds$$

is a zero mean martingale whose variance is $O(T)$. Thus Kronecker's Lemma implies that $M^{\Delta}(T)/T \to 0$ w.p.1. This implies that for the purpose of evaluating the limit of the left side of (7.12), we can replace it by

$$\frac{1}{T}\int_0^T q(x^{\Delta}(s))ds, \tag{7.13}$$

where we define

$$q(x^{\Delta}(s)) = E^{\alpha_0}_{x^{\Delta}(s)} Y^{\Delta}(t_0, s) = \frac{d}{d\alpha} E^{\alpha}_{x^{\Delta}(s)} f(x^{\Delta}(t_0 + s))\big|_{\alpha=\alpha_0}.$$

The function $q(\cdot)$ is continuous and bounded. Then, the ergodic properties of $\{X^{\Delta}_n, n < \infty\}$ imply that (as $T \to \infty$) (7.13) converges w.p.1. to its mean value $\langle \mu^{\Delta}(\alpha_0), q \rangle$ which is just the center term of (7.12). Q.E.D.

8. Numerical Comparisons.

The approximation method of Section 7 has been simulated and compared with alternative methods on a variety of problems of dimension up to seven. Here, we comment on some comparisons with a finite difference method. The alternative methods are all described and discussed in [1], and we will repeat only a few of the comments made there.

The basic method used for all methods takes one long simulation, over an interval T_1. A basic estimation interval T_0 is given, and the approximate model $X^\Delta(\cdot)$ is simulated. $N = T_1/T_0$ estimates of the derivative are made in the long simulation interval, each using T_0 units of time. Let X_n^Δ denote the state of the system at the start of the n^{th}subinterval. Then X_n^Δ is the initial condition for the estimate on the $(n+1)^{st}$ subinterval. The detailed results reported here are for a two dimensional problem, with the parameter α being a scalar. We comment on larger problems later. For the finite difference estimate, a pair of simulations must be taken, with a parameter set at $\alpha_0 \pm \delta\alpha$, for some small $\delta\alpha$. The samples of the δw in (5.1) for the second member of the pair was the same as that of the first member of the pair, with the samples being independent from pair to pair. This reduced the variance over what would have been the case if all the samples of the δw random variables has been mutually independent, as in [1]. The reduction was particularly large if the system was linear, and the cost function smooth, although there was a noticeable reduction in the variance in all cases tested.

The two dimensional problem was the noise driven Van der Pol equation

$$dx_1 = x_2 dt$$

$$dx_2 = [10x_2(1 - x_1^2) - \alpha x_1]dt + dw,$$

where $\alpha_0 = 2$. Note that this system is degenerate. Nevertheless the method works well. The cost function of interest was

$$\int_0^S k(x(s))ds/S$$

for large S, where

$$k(x) = I_{\{|x_2| \geq 0.3\}}.$$

The simplest estimator is

$$\frac{1}{N}\sum_{n=1}^{N}\frac{1}{T_0}\int_{nT_0}^{nT_0+T_0}[Z^\Delta(s,\alpha_0) - Z^\Delta(nT_0,\alpha_0)]k(X^\Delta(s))ds. \tag{8.1}$$

An "antithetic" variable method was always used since it gives a reduced variance: Let N be an even number, and let the δw samples used for the $2n$'th estimate be the negative of that used for the $2n-1$'th $(n = 1, 2, \cdots)$ estimate, with the δw used for the $2n-1$'th estimates $(n = 1, 2, \cdots, N/2)$ being mutually independent.

The centered form, where $k(X^\Delta(s))$ is replaced by the centered $k(X^\Delta(s)) - \bar{k}(nT_0 + T_0)$, where the centering is a sample estimate of the value of the cost at the cited time, actually gave better results. This method is referred to as the AC-method in the tables below (antithetic variable, centered). The centering is zero mean, but helps reduce the variance. As $n \to \infty$, (8.1) converges to

$$\frac{d}{d\alpha}\int \mu(dx,\alpha_0)E_x^{\alpha_0}k(X^\Delta(T_0)).$$

For large enough T_0, this is a good estimate of the desired derivative. A better procedure would be to divide the interval $[0, T_0]$ into a reasonable number of subintervals to get a better approximation to the first centered form discussed in Section 5. But one must keep in mind that the CPU time required for a large number of subdivisions might be better used for taking more samples.

A third method, called the weighted AC-method, often (but not always) was advantageous. As $s \to \infty$, the variance of $[Z^{\Delta}(s, \alpha_0) - Z^{\Delta}(nT_0, \alpha_0)]$ goes to ∞. If the system has a "short" memory, then the "earlier" part of the $Z^{\Delta}(\cdot)$ process contributes little to the estimate in the following sense: Let $nT_0 + T_0 > s > s_0 > nT_0$, and write

$$[Z^{\Delta}(s, \alpha_0) - Z^{\Delta}(nT_0, \alpha_0)]k(X^{\Delta}(s)) =$$

$$[Z^{\Delta}(s, \alpha_0) - Z^{\Delta}(s_0, \alpha_0)]k(X^{\Delta}(s)) +$$

$$[Z^{\Delta}(s_0, \alpha_0) - Z^{\Delta}(nT_0, \alpha_0)]k(X^{\Delta}(s)).$$

Then the mean value of the second term goes to zero as $s - s_0 \to \infty$. But, if we reduce the sample interval, then a bias is added. In order to balance the opposing effects, we use a weighted substitute $\tilde{Z}^{\Delta}$ for Z^{Δ}, constructed as follows, where $\lambda \in (0,1)$ is a weighing factor or exponential discount of the past: (notation for the non-degenerate case);

$$\tilde{Z}^{\Delta}((n+1)\Delta) = [\sigma^{-1}(X_i^{\Delta}, \alpha_0) b_{\alpha}(X_i^{\Delta}, \alpha_0)]' \delta w(i\Delta) - \lambda \tilde{Z}^{\Delta}(n\Delta).$$

For the problem reported on here, this method gave excellent results. In other cases, where the "approach to ergodicity" is slower, a substantial bias could be introduced into the estimates.

Refer to the tables, where the sample means of the derivative estimates, their sample standard deviations, and the required CPU time are given. For the finite difference estimates, N=2,500 was used, and N=5,000 otherwise. This is because two system simulations per finite difference estimate are needed, and only one for our method. But the important quantity is the sample standard deviation per CPU time unit. Note that the sample standard deviation for the weighted AC-method decreases as T_0 increases, while that for the AC method increases. We can readily see the advantages of the methods introduced here. For linear systems, the finite difference method seems to work better owing to the 'smoothness' of the dependence of the estimates on the noise, and the value of the difference interval was not too important (did not seriously affect the sample variance), as long as it was small enough to control the bias.

There are important dimensionality advantages to our methods. Suppose that the dimension of the parameter is m. Then, in order to get a single estimate of a gradient, a finite difference method needs to simulate the system either $(m+1)$ or $2m$ times, depending on the finite difference method used (one sided or central). Our method requires the simulation of only one sample path per estimate, and the calculation of one Z-variable per component of the parameter. But, the calculation of the Z-variable is usually much simpler than doing a simulation of the system. This is particularly true if the system is of high dimension, or if the dynamical terms are hard to compute. Thus, our methods do require much less computer time than does the finite difference method, particularly for high dimensional and nonlinear problems. Alternative methods, such as the finite difference method, can compensate for this only by having a better quality estimate; i.e., one with smaller bias or sample variance.

We emphasize that no general rule has been found which can tell us which method would be preferable for any particular class of problems. All methods must be taken as serious candidates, and techniques sought for their realization so that they perform as well as possible.

$T_0 = 3$

Finite Difference ($\delta\alpha = .05$)

	sample mean	sample standard deviation
derivative	.168	.247
cost	.363	.149
CPU Time		32.04

AC

	sample mean	sample standard deviation
derivative	.164	.216
cost	.364	.127
CPU Time		18.9

Weighted AC (Derivative only)

	sample mean	sample standard deviation
λ		
.1	.160	.19
.5	.153	.14
CPU Time		20.1

TABLE 1

$T_0 = 10$

Finite Difference ($\delta\alpha = .05$)

	sample mean	sample standard deviation
derivative	.157	.243
cost	.364	.052
CPU Time		104.8

AC

	sample mean	sample standard deviation
derivative	.162	.304
cost	.364	.032
CPU Time		65.5

Weighted AC (Derivative only)

	sample mean	sample standard deviation
$\lambda = .5$	.157	.106
$\lambda = 1$	.150	.07
CPU Time		68.3

TABLE 2

$T_0 = 20$

Finite Difference ($\delta\alpha = .05$)

	sample mean	sample standard deviation
derivative	.168	.246
cost	.365	.032
CPU Time		209.8

AC

	sample mean	sample standard deviation
derivative	.168	.537
cost	.365	.021
CPU Time		65.5

Weighted AC (Derivative only)

	sample mean	sample standard deviation
$\lambda = .5$	.154	.058
$\lambda = 1$	.163	.101
CPU Time		137.05

TABLE 3

References

[1] Jichuan Yang, Harold J. Kushner, "A Monte-Carlo method for the sensitivity analysis and parametric optimization of nonlinear stochastic systems', LCDS report 89-90, Brown University. To appear, SIAM J. on Control and Optimization.

[2] R. Z. Khazminskii, "Ergodic properties of recurrant diffusions and stabilization of the solution to the cauchy problem for parabolic systems", Theory Prob. and Applications, **5**, 1960, 179-196 (English translation).

[3] H.J. Kushner, 1978, "Optimality conditions for the average cost per unit time problem with a diffusion model", SIAM J. on Control and Optimization, **16**, 330-346.

[4] S. Orey, *Limit Theorems for Markov Chain Transition Probabilities*, Van Nostrand, London, 1971.

[5] H.J. Kushner, 1984, *Approximation and Weak Convergence Methods for Random Processes, with Applications to Stochastic Systems Theory*. M.I.T. Press, Cambridge, Mass.

[6] S.N. Ethier, T.G. Kurtz, 1986, *Markov Processes: Characterization and Convergence*, Wiley, New York.